Suggested Solutions of Assignment 4:

(i) Since 0 is a regular value of f, by implicit function theorem, $\exists$ a neighborhood of (a,b), say $U \subset \mathbb{R}^2$ s.t $z(x,y) : U \to \mathbb{R}$ is a function. We consider the Monge patch $X(x,y) = (x, y, z(x,y))$

$$X_x = (1, 0, z_x), \quad X_y = (0, 1, z_y)$$

And we have $\quad f(x,y,z) = 0 \quad$ (We denote derivative on the ith position of f, f_i)

$$\begin{cases} f_1 + f_3 \cdot z_x = 0 \\ f_2 + f_3 \cdot z_y = 0 \end{cases} \Rightarrow \begin{cases} z_x = -\dfrac{f_1}{f_3} \\ z_y = -\dfrac{f_2}{f_3} \end{cases} \quad \text{note that } f_3 \neq 0.$$

$$X_x \times X_y = (-z_x, -z_y, 1) = \left(\frac{f_1}{f_3}, \frac{f_2}{f_3}, 1\right)$$

The tangent plane equation at (a,b,c)

$$\left\langle (x,y,z) - (a,b,c), \left(\frac{f_1}{f_3}, \frac{f_2}{f_3}, 1\right)\Big|_{(a,b,c)} \right\rangle = 0.$$

i.e

$$\frac{f_1}{f_3} x + \frac{f_2}{f_3} y + z = a\frac{f_1}{f_3} + b\frac{f_2}{f_3} + c$$

i.e $\quad f_1(a,b,c)\, x + f_2(a,b,c)\, y + f_3(a,b,c)\, z = a f_1(a,b,c) + b f_2(a,b,c) + c f_3(a$

a unit norm $= \dfrac{\frac{1}{f_3}(f_1, f_2, f_3)}{|\frac{1}{f_3}|\sqrt{f_1^2 + f_2^2 + f_3^2}} = (\pm) \dfrac{(f_1, f_2, f_3)}{\sqrt{f_1^2 + f_2^2 + f_3^2}}\Bigg|_{(a,b,c)}$.

(ii) Now $f(x,y,z) = x^2 + y^2 - z^2 - 1$

$f_1 = 2x, \quad f_2 = 2y, \quad f_3 = -2z \quad$ (So $\nabla f \neq 0 \ \forall (x,y,z)$ s.t $f(x,y,z) =$

So the tangent plane equation at $(a,b,0)$

$$2ax + 2by = a(2a) + b(2b)$$

i.e

$$ax + by = a^2 + b^2 = 1$$

i.e $ax + by = 1$.

a unit normal $= \dfrac{(2a, 2b, 0)}{\sqrt{4a^2+4b^2}} = \dfrac{(a, b, 0)}{\sqrt{a^2+b^2}} = (a, b, 0)$.

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(2) (i) $X_u = (a\cos u\cos v,\ b\cos u\sin v,\ -c\sin u)$

$X_v = (-a\sin u\sin v,\ b\sin u\cos v,\ 0)$

$E = \langle X_u, X_u\rangle = a^2\cos^2 u\cos^2 v + b^2\cos^2 u\sin^2 v + c^2\sin^2 u$

$F = \langle X_u, X_v\rangle = -a^2\sin u\cos u\sin v\cos v + b^2\sin u\cos u\sin v\cos v$

$= (b^2 - a^2)\sin u\cos u\sin v\cos v$

$G = \langle X_v, X_v\rangle = a^2\sin^2 u\sin^2 v + b^2\sin^2 u\cos^2 v.$

(ii) $X_u = \left(\dfrac{4(v^2-u^2+4)}{(u^2+v^2+4)^2}, \dfrac{-8uv}{(u^2+v^2+4)^2}, \dfrac{16u}{(u^2+v^2+4)^2}\right)$

$X_v = \left(\dfrac{4(u^2-v^2+4)}{(u^2+v^2+4)^2}, \dfrac{-8uv}{(u^2+v^2+4)^2}, \dfrac{16v}{(u^2+v^2+4)^2}\right)$

$E = \dfrac{1}{(u^2+v^2+4)^4}\left(16(v^4+u^4+16+8v^2-8u^2-2u^2v^2) + 64u^2v^2 + 256u^2\right)$

$= \dfrac{16(u^2+v^2+4)^2}{(u^2+v^2+4)^4} = \dfrac{16}{(u^2+v^2+4)^2}.$

$$F = \frac{1}{(u^2+v^2+4)^4}\left((-8uv)\cdot 4(v^2-u^2+4) + (-8uv)4(u^2-v^2+4) + 256uv \quad \cancel{16(2u^2v^2 - v^4 - u^4 + 16) + 64u^2v^2 + 256uv} \right)$$

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$$= 0 \quad \cancel{\frac{16}{(u^2+v^2+4)^4}\cdot\left(6u^2v^2 - v^4 - u^4 + 16uv\right)}.$$

$$G = \frac{16}{(u^2+v^2+4)^2}$$

(3) This is a general fact for cross product and dot product.

$$|a\times b|^2 = |a|^2|b|^2 - \langle a,b\rangle^2$$

$$\text{LHS} \triangleq |a|^2|b|^2\sin^2\theta \qquad \theta = \text{angle between } a \text{ and } b.$$

$$= |a|^2|b|^2(1-\cos^2\theta)$$

$$= |a|^2|b|^2 - \langle a,b\rangle^2 \qquad \text{Since } a\cdot b \triangleq |a|\cdot|b|\cos\theta.$$

$$= \text{RHS}.$$

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(4) M may be represented by $X(s,v) = (x(s)\cos v,\ x(s)\sin v,\ z(s))$,

$0 \le s \le l,\quad 0 \le v \le 2\pi$

$$X_s = (x'\cos v,\ x'\sin v,\ z')$$

$$X_v = (-x\sin v,\ x\cos v,\ 0)$$

$$X_s \times X_v = (-z' x\cos v,\ -z' x\sin v,\ x\cdot x')$$

$$|X_s\times X_v| = x\sqrt{x'^2 + z'^2} = x \quad (x>0,\ x(s) \text{ is parametrized by arc-length})$$

$$\text{Area}(M) = \int_0^{2\pi}\int_0^l |X_s\times X_v|\,ds\,dv = 2\pi\int_0^l x(s)\,ds.$$

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